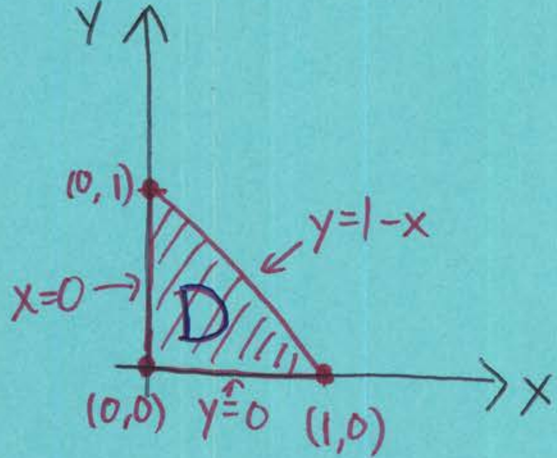
Lecture 24

15.3 - Double Integrals over General Regions

On Monday, I left you with the question of how to integrate the function $z=5$ over the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.



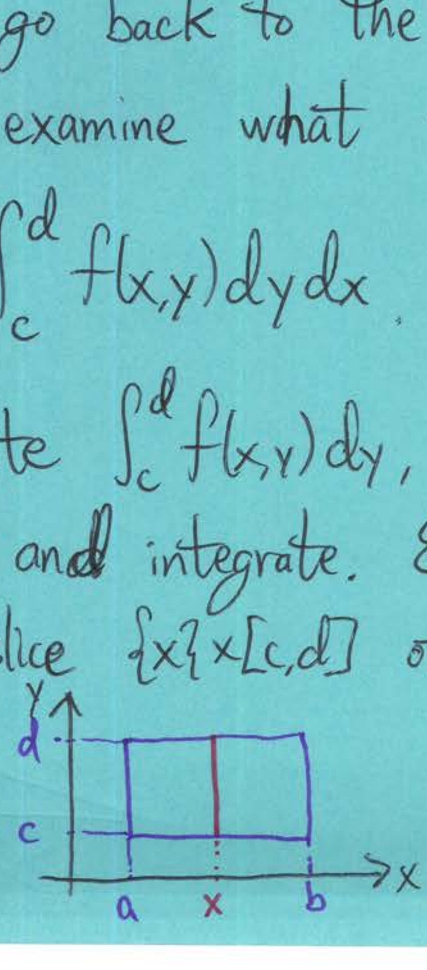
That is, compute the integral $\iint_D 5\,dA$, where D is the triangular region.

If we go back to the case of a rectangle, let us carefully examine what we are doing there:

e.g. $\int_a^b\int_c^d f(x,y)\,dy\,dx$.

To compute $\int_c^d f(x,y)\,dy$, we fix an x value (hold x constant) and integrate. Essentially, this amounts to integrating over the slice $\{x\}\times[c,d]$ of $[a,b]\times[c,d]$.

Visually,

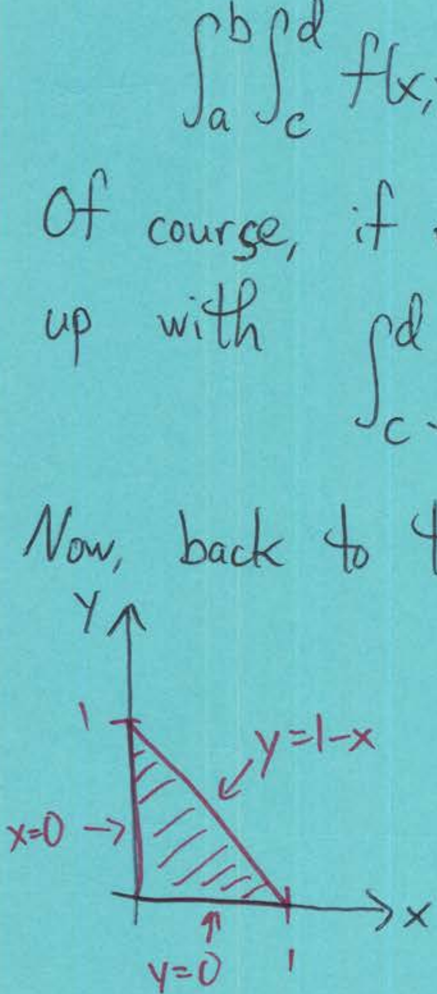


we've integrated over the red line. Now, we need to "add up" the integrals over each such line.

This is done by integrating over x, hence we have
$$\int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Of course, if we take slices the other way, we end up with
$$\int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Now, back to the triangle:



If we take vertical slices, i.e. hold x constant, then y varies between 0 and $1-x$ (remember, x is a fixed constant for the moment), so to integrate over this slice, we have: $\int_0^{1-x} 5\,dy$. Now, we add up all these vertical slices:
$$\iint_D 5\,dA = \int_0^1 \int_0^{1-x} 5\,dy\,dx.$$

The bounds on the x integral are 0 to 1 since those are the values x takes on in D. So then:
$$\iint_D 5\,dA = \int_0^1 \int_0^{1-x} 5\,dy\,dx = \int_0^1 \left(5y\Big|_0^{1-x}\right) dx = \int_0^1 (5(1-x)-0)\,dx$$
$$= \int_0^1 (5-5x)\,dx = \left(5x - \tfrac{5}{2}x\right)\Big|_0^1 = \left(5-\tfrac{5}{2}\right) - 0 = \frac{5}{2}$$

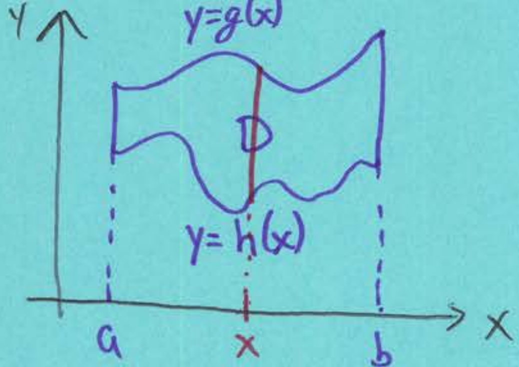
Some Comments:

1) The outside integral should <u>NEVER</u> have a variable in it. It can have an arbitrary constant in it, but never a variable which is being integrated over.

2) I <u>highly recommend</u> ALWAYS sketching the region you are integrating over. It really helps you set up the bounds.

3) To find the bounds, first draw your region, then decide which way you want to take your slices:

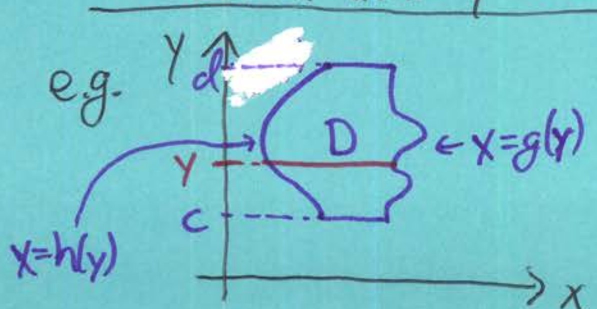
- <u>vertical (hold x constant)</u>: you look from the bottom to the top: e.g.,



$$\iint_D f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx$$

- <u>horizontal (hold y constant)</u>: look from left to right:

e.g.



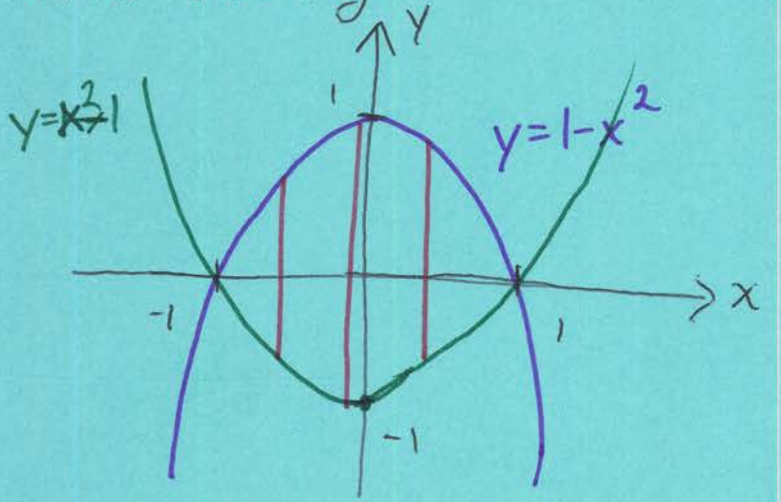
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dx\,dy$$

4) Sometimes integrating one way is easier than another.

Let's do examples now:

Ex: Compute $\iint_D x\,dA$ where D is the region bounded by the parabolas $y=1-x^2$ and $y=x^2-1$.

Sol: First, draw the region:



Vertical slices work nicely here:

$$\iint_D x\,dA = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} x\,dy\,dx = \int_{-1}^{1} (xy)\Big|_{x^2-1}^{1-x^2} dx$$

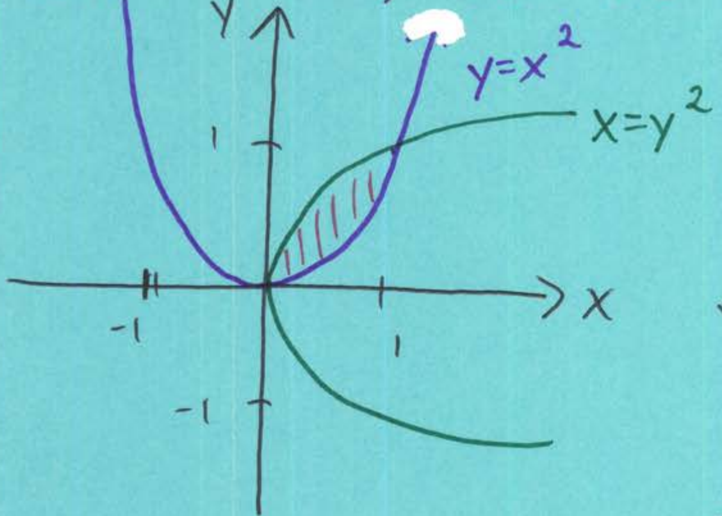
$$= \int_{-1}^{1} \left[x(1-x^2) - x(x^2-1)\right] dx = \int_{-1}^{1} (x^2-x^3-x^3+x^2)\,dx$$

$$= \int_{-1}^{1} (2x^2-2x^3)\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^4\Big|_{-1}^{1} = \left(\frac{2}{3}-\frac{1}{2}\right) - \left(-\frac{2}{3}-\frac{1}{2}\right) = \frac{4}{3}$$ ◇

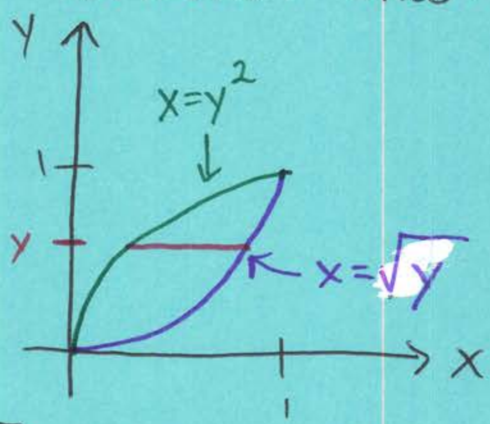
From now on, I'll focus more on the set up of an integral, rather than its computation.

Ex: Set up the integral of $f(x,y) = \sin(xy)$ over the region D where D is the region bounded by $y = x^2$ and $x = y^2$.

Sol: As usual, draw the region first



We can take slices of this in either direction. Let's take horizontal ones:



So, $\iint_D \sin(xy)\, dA = \int_0^1 \int_{y^2}^{\sqrt{y}} \sin(xy)\, dx\, dy$

$\diamond$

Ex: Set up the integral to find the area of the region bounded by the lines $y=0$, $y=x$, and $y=4-2x$.

Sol: Loosely speaking, the volume of a solid is the area of the base, (if the base is a region D, its area is $A(D)$) times the height. If the height changes according to a function $f(x,y)$ on D, the volume is $V = \iint_D f(x,y)\, dA$.

If a solid has height always 1, then

$$\text{Vol} = (\text{area of base})(\text{height}) = (A(D))(1) = A(D).$$

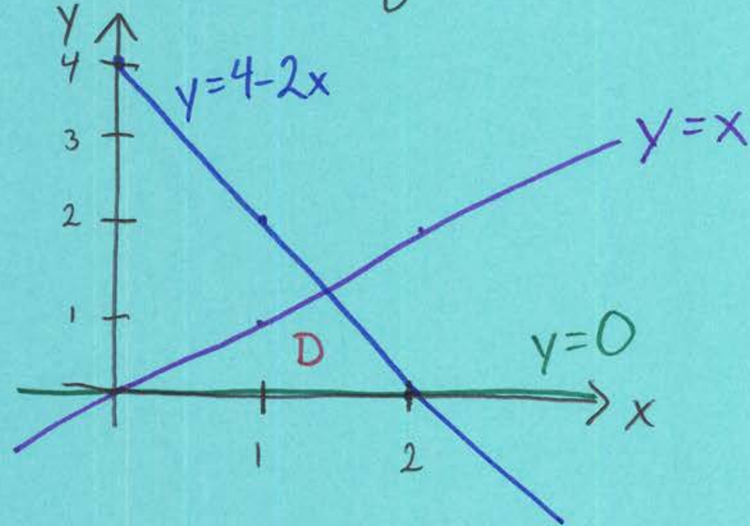
In terms of integrals:

$$\text{Vol} = \iint_D f(x,y)\,dA = \iint_D 1\,dA = A(D).$$

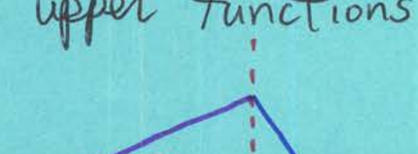
So, the area of a region D is given by

$$\boxed{A(D) = \iint_D dA.}$$

Let's draw the region D:



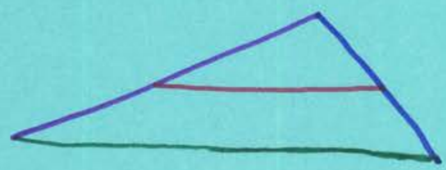
Again, we could do this integral with vertical or horizontal slices, but one requires more work. If we do vertical slices, we have to split the region in two according to the two upper functions:



So

$$A(D) = A(D_1) + A(D_2)$$

However, if we use horizontal slices, this problem doesn't exist.

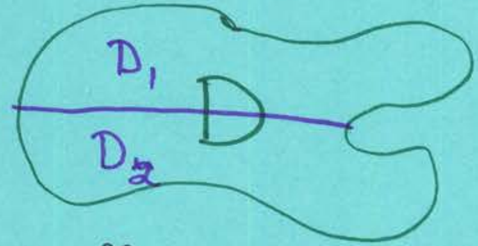


The bounds on the inside integral are y to $\frac{4-y}{2}$. On the outside integral, the lower bound is 0, and the upper bound is the y-value of the intersection of $y=x$ with $y=4-2x$. To solve this, simply plug one equation into the other: $y=x$ into $y=4-2x \Rightarrow y=4-2y \Rightarrow 3y=4 \Rightarrow y=\frac{4}{3}$. So, the area integral is:

$$A(D)=\int_0^{4/3}\int_y^{\frac{4-y}{2}} dx\,dy.$$

$\diamond$

As briefly explained in this example, if we can write D as the union of two regions, $D=D_1 \cup D_2$, e.g.:



then $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$

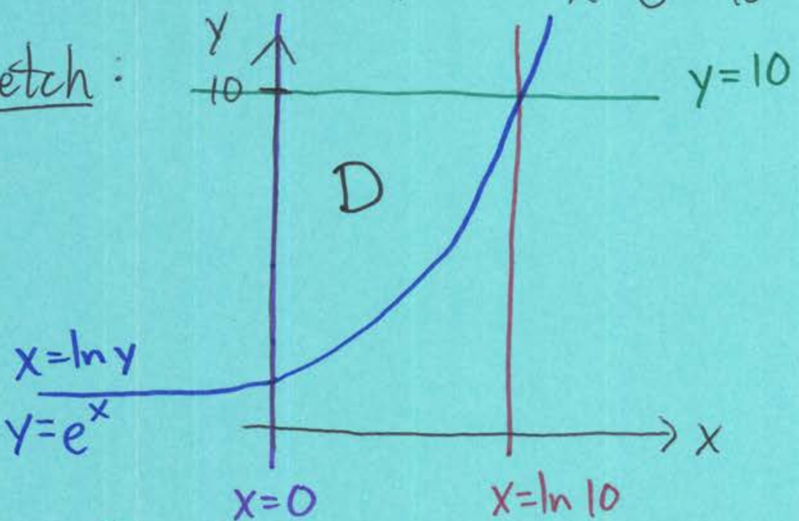
The most important case I can make for always sketching the region of integration is the following. This shows that you should be able to sketch the region given the integral itself.

<u>Ex</u>: Compute the double integral $\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx$.

<u>Sol</u>: How do you compute $\int \frac{1}{\ln y}\,dy$? Who knows!

The region of integration is given by the bounds of the integral: the inside integral goes from $y=e^x$ to $y=10$ and the outside from $x=0$ to $x=\ln 10$.

<u>Sketch</u>:



Using this, we can set up the integral with dxdy:

$$\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx = \int_1^{10}\int_0^{\ln y} \frac{1}{\ln y}\,dx\,dy = \int_1^{10}\left(\frac{x}{\ln y}\Big|_0^{\ln y}\right)dy = \int_1^{10} 1\,dy$$

$$= y\Big|_1^{10} = 9.$$

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